OVIDIU GHERASIM
IOANA DORNESCU

MAKING DECISIONS BY USING FUZZY MODELS

1. The general form of the decision adoption models

In the most specialty papers, a mathematical model for establishing the best decisional alternatives is a couple M=(R, K) made up of a matrix R with m lines and n columns and a column vector K with m components.

***The results matrix* R** and the ***importance coefficients vector* K** are thus tabulated:

Table 1

The results matrix and the importance coefficients vector

	V_1	V_2	…	V_j	…	V_n	K
C_1	R_{11}	R_{12}	…	R_{1j}	…	R_{1n}	k_1
C_2	R_{21}	R_{22}	…	R_{2j}	…	R_{2n}	k_2
…	…	…	…	…	…	…	…
C_i	R_{i1}	R_{i2}	…	R_{ij}	…	R_{in}	k_i
…	…	…	…	…	…	…	…
C_m	R_{m1}	R_{m2}	…	R_{mj}	…	R_{mn}	k_m

The n columns V_j of the R matrix represent the ***decisional alternatives*** that we wish to arrange.

The m lines C_i are the ***criteria***, objectives or the nature‘s states.

The generic element R_{ij} is the result of C_i criteria obtained when the V_j decisional alternative is chosen.

The criteria may be of two types:

- ***criteria of profit*** type (of maximum) when the bigger the result of the criteria the best the decisional alternative (big profit);
- ***criteria of cost type*** (of minimum) when the smaller the result of the criteria the best the decisional alternative (small cost).

The group of experts gives a note N_i, to each criterion according to the importance of that criterion.

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To transform the notes into weights (positive sub-unitary numbers with unitary sum), the following relation is used:

$$k_i = \frac{N_i}{\sum_{p=1}^{m} N_p}, \quad i = \overline{1,m} \quad \Rightarrow \quad \sum_{i=1}^{m} k_i = 1 \tag{1}$$

The m ***importance coefficients*** k_i form the K vector.

2. The method of global utilities and the ELECTRE method of ranking the decisional alternatives

A ranking of the n decisional alternatives V_j is equivalent to arranging the columns of the results matrix R.

Because on each of the two different lines of the matrix there are totally different values, expressed through different measures, the comparing can be done if we insert these values in the [0, 1] interval.

On each line the smallest and the greatest element earns the 0 and 1 ,for the profit type criterion (of maximum), and 1 and 0, for the cost type criterion (of minimum).

The linear insertion assures the maintenance of the proportionality between utilities and the initial results and may be done using the following relations:

$$u_{ij}^{min} = \frac{R_i^{max} - R_{ij}}{R_i^{max} - R_i^{min}}, \quad u_{ij}^{max} = 1 - u_{ij}^{min} = \frac{R_{ij} - R_i^{min}}{R_i^{max} - R_i^{min}} \tag{2}$$

where: $R_i^{max} = \max\limits_{1 \le j \le n} R_{ij}$ and $R_i^{min} = \min\limits_{1 \le j \le n} R_{ij}$.

$$u_{ij} = \begin{cases} u_{ij}^{min} & \text{for } C_i \text{ minimum criteria} \\ u_{ij}^{max} = 1 - u_{ij}^{min} & \text{for } C_i \text{ maximum criteria} \end{cases} \tag{3}$$

The columns of the utilities matrix U may be now compared because the elements of this matrix are positive numbers, sub-unitary or unitary, amorphous (without measure)

Therefore, the ***utilities of the alternatives*** must be calculated as ponderated average of all the m criteria utilities:

$$U_j = U(V_j) = \sum_{i=1}^{m} k_i \cdot u_{ij} \tag{4}$$

The order of the alternatives' utilities introduces the order between alternatives.

Thus the ***V_p alternative outruns the V_q alternative*** if the U_p utility is bigger than the U_q utility:

$$V_p > V_q \quad \Leftrightarrow \quad U_p > U_q \tag{5}$$

Therefore, the best decisional alternative V*, obtained by the global utilities method, is the alternative to which the greater utility corresponds:

$$V^* = V_{j0} \quad \Leftrightarrow \quad U_{j0} = \max_{1 \le j \le n} U_j \tag{6}$$

The basis of ELECTRE method (Elimination Et Choix Traduisant la Realite) were put in 1965 by a group of French researchers from SEMA (Société de l'Economie et des Mathematiques Appliqués).

This method requires first, the calculation of two groups of indicators for all the alternative pairs: ***the concordance indicators*** and ***the discordance indicators..***

The concordance indicator $\mathbf{C_{pq}}$ between the alternatives V_p and V_q coincides to the ***discordance indicator*** $\mathbf{D_{qp}}$ between the V_q and V_p alternatives and represents the weighted sum of all the positive differences between the utilities of the alternatives:

$$c_{pq} = d_{qp} = \sum_{\substack{i=1 \\ u_{ip} > u_{iq}}}^{m} k_i \cdot (u_{ip} - u_{iq}) \qquad , p, q = \overline{1, n} \tag{7}$$

The matrix of all concordance indicators $C = \left(c_{pq}\right)_{p,q=\overline{1,n}}$ is called ***the concordances matrix*** and its transposed $C^T = D = \left(d_{pq}\right)_{p,q=\overline{1,n}}$ is called ***the discordances matrix***

With the help of the concordance and discordance indicators we establish a relation of outrunning the alternatives.

The V_p alternative outruns the V_q alternative if the concordance indicator dominates the discordance indicator of the two alternatives:

$$V_p > V_q \quad \Leftrightarrow \quad c_{pq} > d_{pq} (= c_{qp}) \tag{8}$$

The binary matrix of the outrunning $\left(b_{pq}\right)_{p,q=\overline{1,n}}$ can be obtained by setting on 1 the positive elements of the matrix obtained as difference between the concordance and discordance matrix C–D, and its elements are thus defined:

$$b_{pq} = \begin{cases} 1 & , c_{pq} > c_{qp} \\ 0 & , c_{pq} \le c_{qp} \end{cases} \tag{9}$$

This matrix corresponds to a graph named ***the graph of the outrunning.***

The graph of the outrunning has an arch orientated from the V_p knot to the V_q knot if the element of the p line and q column from the binary matrix is 1.

By summing the columns of the outrunning binary matrix it is obtained the ***vector of the outrunning*** **(**column vector**)** $\left(s_p\right)_{p=\overline{1,n}}$ where the elements represent ***the number of outrunning of each alternative*** V_j over the other alternatives:

$$s_p = \sum_{q=1}^{n} b_{pq} \quad , p = \overline{1,n} \tag{10}$$

The best decisional alternative V^* obtained by ELECTRE method is the alternative with the bigger number of outrunning:

$$V^* = V_{p_0} \quad \Leftrightarrow \quad s_{p_0} = \max_{1 \le p \le n} s_p \tag{11}$$

3. The mathematical modeling of the uncertain information using Fuzzy triangular numbers and their operations

The mathematical modeling of an uncertain information evaluates its most probable value a_m and the minus possible values a_s and plus a_d.

The three real values form an ordered triplet $\tilde{a} = (a_s, a_m, a_d)$ named ***fuzzy triangular number***.

For example, let's suppose the unitary price of a raw material is 800 lei. Considering the medium inflation and its fluctuations, one can express the probable price of the raw material for the next year, by the fuzzy triangular number $\widetilde{p} = (850, 900, 1000)$.

As concerning two triangular fuzzy numbers associated real numbers can be defined, the multiplication by scalar (real number), the four arithmetical operations, and an order relation, by the following relations:

$\widetilde{a} = (a_s, a_m, a_d)$	$\widetilde{b} = (b_s, b_m, b_d)$
associate real number:	$<\widetilde{a}> = \dfrac{2a_m + a_s + a_d}{4}$
multiplication by scalar:	$t\widetilde{a} = \begin{cases} (ta_s,\ ta_m,\ ta_d) & ,t > 0 \\ (ta_d,\ ta_m,\ ta_s) & ,t < 0 \end{cases}$
addition:	$\widetilde{a} + \widetilde{b} = (a_s + b_s, a_m + b_m, a_d + b_d)$
subtract:	$\widetilde{a} - \widetilde{b} = (a_s - b_d, a_m - b_m, a_d - b_s)$
multiplication:	$\widetilde{a}\widetilde{b} = \dfrac{\widetilde{a}<\widetilde{b}> + <\widetilde{a}>\widetilde{b}}{2}$
division:	$\dfrac{\widetilde{a}}{\widetilde{b}} = \dfrac{\widetilde{a}<\widetilde{b}> + <\widetilde{a}>\widetilde{b}}{2<\widetilde{b}>^2}$
order relation:	$<\widetilde{a}> \underset{(>)}{<} <\widetilde{b}> \Rightarrow \widetilde{a} \underset{(\succ)}{\prec} \widetilde{b}$

(12)

These fuzzy theory notions can be found in many specialty works. Here there are just two of them: “Fuzzy sets, fuzzy logic, application” [1] and “The Mathematics of Triangular Fuzzy Numbers with Applications for the Study of Managerial Decisions” [2].

4. Fuzzy models in taking decisions

The results of the decisional alternatives for each criteria or state of nature (shown in no.1 table) are information that can not be exactly determined in practice.

The size of these information obtained through various tests, simulations and polls, as well as the variation of this determined values, both in minus and plus, suggest the utilization of fuzzy triangular numbers ($\widetilde{R}_{ij}$).

A ***fuzzy model*** for establishing the best decisional alternative is a couple $\widetilde{M} = (\widetilde{R}, K)$ formed by the results matrix having fuzzy triangular numbers as elements and the importance coefficients vector K.

The ***associated classical model*** $<\tilde{M}> = (<\tilde{R}>, K)$ of a fuzzy model is obtained by replacing the results (fuzzy numbers) with associate real numbers.

Because of all the operations in the relations (2)-(11) were defined for fuzzy numbers as well in relations (12), the global utility method and the ELECTRE method for a fuzzy model follows the same path as a classical model only that all the operations in those relations are performed with fuzzy triangular numbers.

These models, (fuzzy and/or classical) are ***equivalent models*** related to a method of hierarchy of decisional alternatives if the two models have the same criteria and alternatives and by using the method to both models, the same hierarchy is obtained.

Theorem. The fuzzy model and the classical associate model are equivalent related to the global utilities method and to ELECTRE method of hierarchy the decisional alternatives.

The demonstration of the theorem it is not to be presented due to the fact that it is too big reported to the length of the article and due to the fact that can be consulted in the paper [2].

To make the way of working easier, I shall present as follows, the calculations for a hypothetical model of reduced dimensions.

Example. Given the initial model with three decisional alternatives V1 V2 V3 and two objectives C_1, C_2, the first of minimum and the letter of maximum, with the weights k_1=0,45 and k_2=0,55, and the results R_{ij} from the table no.2. Which is the best decisional alternative?

Table 2

The initial model

	V_1	V_2	V_3	k
C_1	128	158	168	0.45
C_2	49	81	89	0.55

Let us apply to this classical model the two methods of establishing the best decisional alternative.

The decisional utilities are calculated according to the relations (2) and (3):

$$u_{11} = \frac{168-128}{168-128} = 1; \quad u_{12} = \frac{168-158}{168-128} = \frac{1}{4} = 0.25; \quad \ldots$$

By using the relations (4) the utilities of the alternatives are obtained:

$$U_1 = 0.45 \cdot 1 + 0.55 \cdot 0 = 0{,}45; \ \ldots$$

All these results are concentrated in the following table.

Table 3

The utilities matrix for the initial model

	V_1	V_2	V_3	k
u_{1j}	1	0.25	0	0.45
u_{2j}	0	0.8	1	0.55
U_j	0.45	0.5525	0.55	

According to the relations (5), the order of the global utilities of the alternatives induces the hierarchy of the alternatives, and with the relations (6) the best decisional method is determined:

$$V_2 > V_3 > V_1 \qquad \text{şi} \qquad \boxed{V^* = V_2}.$$

The concordance indicators for the ELECTRE method are obtained by using the relations (7):

$$c_{12} = 0{,}45 \cdot (1 - 0{,}25) = 0{,}3375; \quad c_{21} = 0{,}55 \cdot (0{,}8 - 0) = 0{,}44; \ \ldots$$

The concordance matrix, the binary matrix of the outrunnings and the outrunning vector are obtained by using the relations (9) and (10) :

$$C = \begin{pmatrix} 0 & 0{,}3375 & 0{,}45 \\ 0{,}44 & 0 & 0{,}1125 \\ 0{,}55 & 0{,}11 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \quad S = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

By using the relations (11) the hierarchy of the alternatives is obtained as well as the best decisional alternative for the ELECTRE method:

$$V_2 > V_3 > V_1 \qquad \text{şi} \qquad \boxed{V^* = V_2}.$$

It can be observed that in that particular case by applying the both methods, the same hierarchy was obtained.

The determined is not at all assured of this information that is based on certain simulations, tests, etc.

Because of this reason, a maximum ±20% error is considered for the dates, which leads to a fuzzy model with the entrance dates in the table no.4.

The incertitude of the entrance information (the results) was shaped by fuzzy triangular numbers to reduce the size of the calculation. This fact does not presume the impossibility of using other types of fuzzy numbers in shaping: rectangular, square, etc.

Table 4

The fuzzy model

	V_1	V_2	V_3	K
C_1	(120; 128; 144)	(150; 158; 174)	(152; 168; 176)	0.45
C_2	(47; 49; 59)	(77; 81; 89)	(79; 89; 91)	0.55

On a first stage we obtain the classical associate model by calculating the real numbers associated to the entrance dates using the first definition from the relations (12):

$$\langle 120;128;144\rangle = \frac{2\cdot 128+120+144}{4} = \frac{256+264}{4} = 130;\ \ldots$$

Further on we shall do all the calculations simultaneously both for the fuzzy model and the associate model.

The decisional utilities are calculated using the (2) and (3) relations, the calculations with fuzzy numbers being made according to the relations (12) :

$$\tilde{u}_{11} = \frac{(152;168;176)-(120;128;144)}{(152;168;176)-(120;128;144)} = \frac{(8;40;56)}{(8;40;56)} =$$

$$= \frac{<8;40;56>\cdot(8;40;56)+(8;40;56)\cdot<8;40;56>}{2\cdot<8;40;56>^2} =$$

$$= \frac{2\cdot 36\cdot(8;40;56)}{2\cdot 36^2} = \frac{(8;40;56)}{36} = \frac{(2;10;14)}{9};$$

$$<\tilde{u}_{11}> = \frac{166-130}{166-130} = \frac{36}{36} = 1;\ \ldots$$

All these results are synthesized in the following table:

Table 5

	The Fuzzy Model				The Associate Model		
	V_1	V_2	V_3	k	V_1	V_2	V_3
C_1	(120;128;144)	(150;158;174)	(152;168;176)	0.45	130	160	166
C_2	(47;49;59)	(77;81;89)	(79;89;91)	0.55	51	82	87
$\tilde{u}_{1j}$	$\frac{(2;10;14)}{9}$	$\frac{(-31;25;53)}{108}$	$\frac{(-1;0;1)}{3}$	0.45	1	$\frac{1}{6}$	0
$\tilde{u}_{2j}$	$\frac{(-1;0;1)}{6}$	$\frac{(317;598;719)}{18\cdot 36}$	$\frac{(5;10;11)}{9}$	0.55	0	$\frac{31}{36}$	1
$\tilde{U}_j$	$\frac{(1;60;95)}{120}$	$\frac{(1813;7928;10771)}{12960}$	$\frac{(23;55;65)}{90}$		0.45	$\frac{79}{144}$	0.55

The global utilities in the last row of the table are solved with the relations (4):

$$\widetilde{U}_1 = \frac{9}{20}\cdot\frac{(2;10;14)}{9} + \frac{11}{20}\cdot\frac{(-1;0;1)}{6} =$$

$$= \frac{(12;60;84)+(-11;0;11)}{120} = \frac{(1;60;95)}{120}; \quad <\widetilde{U}_1> = 0.45\cdot 1 + 0.55\cdot 0 = 0.45; \ \ldots$$

The order of the global utility of the alternatives induces the following alternatives' hierarchy:

$$V_3 > V_2 > V_1 \qquad \text{şi} \qquad \boxed{V^* = V_3}.$$

It can be noticed that the best decisional alternative obtained by the global utility method is now V_3.

The concordance indicators for the ELECTRE method are obtained by using the relations (7):

$$\widetilde{c}_{12} = \frac{9}{20}\cdot\left(\frac{(2;10;14)}{9} - \frac{(-31;25;53)}{108}\right) = \frac{(-29;95;199)}{20\cdot 12};$$

$$<\widetilde{c}_{12}> = \frac{9}{20}\cdot\left(1-\frac{1}{6}\right) = \frac{3}{8} = 0.375; \ \ldots$$

The concordance matrix for the fuzzy models and the associate model are:

$$\widetilde{C} = \begin{pmatrix} 0 & \frac{(-29;95;199)}{20\cdot 12} & \frac{(-1;10;17)}{20} \\ \frac{11\cdot(209;598;827)}{12960} & 0 & \frac{(-34;25;56)}{20\cdot 12} \\ \frac{11\cdot(7;20;25)}{360} & \frac{(-34;25;56)}{20\cdot 12} & 0 \end{pmatrix}$$

$$<\widetilde{C}> = \begin{pmatrix} 0 & 0.375 & 0.45 \\ 0.4736(1) & 0 & 0.075 \\ 0.55 & 0.0763(8) & 0 \end{pmatrix}$$

Results that the binary matrix of the outrunning and the outrunning vectors for the fuzzy mode and the associate model are the same:

$$\widetilde{B} = <\widetilde{B}> = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}; \qquad \widetilde{S} = <\widetilde{S}> = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

The fuzzy and the associate models are equivalent related to ELECTRE method and the hierarchy of the alternatives and the best decisional method for these models are:

$$V_3 > V_2 > V_1 \quad \text{and} \qquad \boxed{V^* = V_3}.$$

5. Conclusions

Both through the given theorem and the given example it comes out that any fuzzy model is equivalent to the classical associated mode reported to both methods: the global utilities method and ELECTRE method.

Because of this, in practice first must be determined the model associated to a fuzzy model, and secondly for this classical model the methods are applied and thus the calculations are considerably reduced.

The best decisional alternative of the associate model is also the best for the fuzzy model.

But the best decisional alternative of the initial mode does not correspond the fuzzy model.

This fact leads to a more ample analysis of the practical problems by taking under consideration the incertitude of the information and its modeling by fuzzy techniques.

References

1.Bojadziev G., Bojadziev M., *Fuzzy sets, fuzzy logic, application*, London, 1995.
2. Gherasim, O., *The Mathematics of Triangular Fuzzy Numbers with Applications for the Study of Managerial Decisions,* Performantica Publishing House, Iaşi, 2005.